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Applications of dynamical systems to immunology and to random exchange economies

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Dedicated to my kids; Yusuf and Halima

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Abstract

This thesis presents the results of research in two different areas of applied mathematics. Firstly, mathematical biology and secondly in mathematical economics.

Mathematical biology: We study an ODE system that models immune responses by $CD4^+$ T cells, with the presence of Regulatory T cells (Tregs). We assume that the secreting T cells have a lower death rate than the non secreting T cells and that the active Tregs also have a lower death rate than the inactive Tregs. We present explicit formulas that give the relation at equilibria between the concentration of T cells, the concentration of Tregs and the antigenic stimulation of T cells. For some parameter values, we found a hysteresis, characterized by a region of bistability, with two stable equilibria and one unstable equilibrium, bounded by two thresholds of antigenic stimulation of T cells. At some parameter values, we observe an unfold of the hysteresis. Moreover, we consider a model with a linear tuning between the antigenic stimulation of T cells and the antigenic stimulation of Tregs. For this model, we also present explicit formulas for the relation at equilibria between the concentration of T cells, the concentration of Tregs and the antigenic stimulation of T cells. In this model, the hysteresis is also present. Furthermore, for some parameter values, we observe the appearance of an isolated region with equilibria, an isola, and for other values of the parameters, we observe a transcritic bifurcation.

Mathematical economics: We study a model of Edgeworthian economies where, at each time period, the participants are randomly matched in pairs to exchange two goods. The participants trade with the aim of increasing their utility, given by Cobb-Douglas utility functions, and are allowed to deviate from their bilateral equilibrium. Provided the market and the trade follow appropriate symmetry conditions, we show that, at each time period, the expectation of the logarithm of the trading price is equal to the expectation of the logarithm of the Walrasian price, being both fixed along the time.

Keywords: Dynamical Systems, Game Theory, Immunity, T cells, Tregs, General Equilibrium, Walrasian price, random matching market.

Resumo

Esta tese apresenta os resultados da investigação em duas áreas diferentes de matemática aplicada. Em primeiro lugar, em biologia matemática e em segundo lugar em economia matemática.

Biologia matemática: Estudamos um sistema de EDO que modela respostas imunitárias por células T $CD4^+$, na presença de células T Reguladoras (Tregs). Assumimos que as células T secretoras têm uma taxa de mortalidade menor do que as células T não secretoras e que as Tregs ativas também têm uma taxa de mortalidade menor do que as Tregs inativas. Apresentamos fórmulas explícitas que dão a relação em equilíbrio entre a concentração de células T, a concentração das Tregs e o estímulo antigénico das células T. Para alguns valores dos parâmetros, encontramos uma histerese, caracterizada por uma região de biestabilidade, com dois equilíbrios estáveis e um equilíbrio instável, limitada por dois limiares de estímulo antigénico das células T. Ademais, consideramos um modelo com um ajuste linear entre o estímulo antigénico das células T e o estímulo antigénico das Tregs. Para este modelo também apresentamos fórmulas explícitas que dão a relação em equilíbrio entre a concentração de células T, a concentração das Tregs e o estímulo antigénico das células T. Neste modelo, a histerese também está presente. Além disso, para alguns valores dos parâmetros, observamos o aparecimento de uma região isolada com equilíbrios, uma isola, e para outros valores dos parâmetros, observamos uma bifurcação transcrítica.

Economia matemática: Estudamos um modelo de economias Edgeworthianas onde, em cada período de tempo, os participantes são selecionados aleatoriamente em pares para trocar dois bens. Os participantes trocam com o objectivo de aumentar a sua utilidade, dada por funções de utilidade de Cobb-Douglas, e podem desviar-se do seu equilíbrio bilateral. Desde que o mercado e as trocas obedeçam às condições de simetria apropriadas, mostramos que, em cada período de tempo, o valor esperado do logaritmo do preço de troca é igual ao valor esperado do logaritmo do preço Walrasiano, sendo ambos fixos ao longo do tempo.

Palavras-chave: Sistemas Dinâmicos, Teoria de Jogos, Imunidade, células T, Tregs, Equilíbrio Geral, preço Walrasiano, mercado de emparelhamento aleatório

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Chapter 1

General Introduction

This thesis is a result of scientific research work in two areas of applied mathematics: mathematical biology and mathematical economics.

In the first part we study immune responses by $CD4^+$T cells, within the presence of regulatory T cells. The model we use is based on the model initially proposed by Burroughs *et al.* [21] with the modifications proposed by Pinto *et al.* [57]. In Pinto *et al.* [57], asymmetric death rates were considered - the active cells have lower death rates than the inactive ones - emulating the presence of memory T cells. Here, we extend the work of Oliveira *et al.* [54], by showing exact formulas for the equilibria. We observe that the relation between the concentration of T cells, the concentration of Tregs and the antigenic stimulation of T cells is a hysteresis. Inside the region bounded by two antigenic thresholds we found two stable equilibria and one unstable equilibria. We observe that the hysteresis can be unfolded for some values of the parameters. Furthermore, we compute numerically the eigenvalues that allow us to classify the obtained equilibria as stable or unstable. Following the model by Pinto *et al.* [57], we also model a positive correlation between the antigenic stimulation of T cells, and the antigenic stimulation of Tregs using a linear tuning. We were able to obtain explicit formulas for the relation between the concentration of T cells, the concentration of Tregs and the antigenic stimulation of T cells, thereby expanding the results in Burroughs *et al.* [23] and Oliveira *et al.* [54]. Depending on the parameter values, we observed an hysteresis and the appearance of an isolated region of equilibria, an isola, that may merge with the hysteresis. At the point of contact, we have a transcritical bifurcation. This area is a joint work with B.M.P.M. Oliveira, Alberto A. Pinto, Isabel P. Figueiredo, and N. J. Burroughs. This work resulted in a submitted paper [55], and two communications in conferences.

In the second part we study a model of Edgeworthian economies where participants meet in random pairs to exchange two goods. Each participant's endowment and preferences follow a probability distribution that satisfies the p-statistical duality condition. The p-statistical duality condition imposes on each participant a dual (mirrow) participant with opposite preference and symmetric endowments, hence guaranteeing the occurrence of trade with probability greater than zero. Each participant seeks to maximize his utility according to the Cobb-Douglas function. We allow trade to deviate from the bilateral equilibrium to a point outside the core, expanding the model in Ferreira *et al.* [32]. Our model has similar bases to the model with "zero intelligent" traders by Gode and Sunder [37]. Our objective is to study the convergence of transaction prices to equilibrium levels when trade is allowed

to occur to a point that does not belong in the core. This part is a joint work with B.M.P.M. Oliveira, Alberto A. Pinto, Athanassios N. Yannacopoulos, and Bärbel F. Finkenstädt. The work resulted in a paper in preparation [56] and seven communications in conferences.

The work is divided into four chapters, followed by bibliography. In Chapter 2 we present exact equilibria for a model of T cells and Tregs. In Chapter 3 we present a model of Edgeworthian economies. In Chapter 4 we present a general conclusion of the research work.

Chapter 2

Exact equilibria for a model of T cells and Tregs

2.1 Introduction

Human immune system can be triggered by pathogen infections - its primary function is the protection of the host from invasion by virus, bacteria or parasites. Lympocytes are a part of immune system that recognizes and responds to specific antigens, they are a subset of the Leukocytes, also known as white blood cells. T cells are a group of Lymphocytes that mature in the thymus, activated under exposure to their specific antigens, and leading to secretion of growth cytokines, predominantly Interleukin 2 (IL-2). The population of T cells consists of different types, each with different immunological functions and phenotypes. However, the immune response of T cells is specific: it opposes the progression of an infection characterized by activating a unique set of antigen receptors (T cell receptors, TCR) on the cell surface. Usually, T cells proliferate rapidly at the maximum expansion rate following the activation of a small number of them by a pathogen. The infection may be removed during this expansion phase, the expansion stops after some time and the number of activated T cells is reduced drastically, while some of them may become memory T cells during this process.
A healthy individual should have his immune system capable of differentiating between cells infected with a pathogen and uninfected cells in order to properly protect the human system. However, this is not always the case: the immune system may fail to differentiate the uninfected cells from infected ones, targeting self-antigens and triggering an autoimmune response, that may cause tissue damage and even death [65]. The onset and development of autoimmunity have been attributed to factors such as environment, age, and genetic predisposition [13, 20]. There are several possible mechanisms of interaction between a pathogen and the immune system, with a popular opinion that autoimmunity can, in many cases, be attributed to "molecular mimicry" where linear peptide epitopes, processed from viral proteins, mimic normal host self-proteins, leading to a cross-reaction of immune response against virus with host cells [30, 40].
Regulatory T cells (Tregs) play a role in limiting these mistakes due to their immune-suppressive ability. They are generated in the thymus under positive selection by self peptides [41]. At a genetic level, the Tregs express Foxp3, a master regulator of the Treg phenotype inducing CD25, CTLA-4 and GITR expression, all correlating with a suppressive phenotype [64]. In the presence of active Tregs

the growth of conventional T cells is inhibited. Part of this growth inhibition is the inhibition of IL-2 secretion by T cells [67, 70]. A delicate fit is needed to have immune responses whenever necessary, whilst keeping autoimmunity controlled. In order to develop immune responses in the presence of Tregs, a higher number of T cells need to be activated. Hence, secretion inhibition manipulates the growth dynamics and effectively increases the quorum threshold - shown to act as a growth modulator through adjustment of a quorum threshold associated with cytokine growth dynamics. Burroughs *et al.* [21, 22] showed that cytokine dependent growth exhibits a quorum T cell population threshold that determines if immune responses develop on activation. Thus Treg induced secretion inhibition can provide a mechanism for tissue specific regulation of the balance between suppression (control) and immune responses, a balance that can be varied at the local tissue level through the regulation of the local active Treg population size.

Some researches examined the consequences of regulatory T cell (Treg) inhibition of interleukine 2 secretion by the use of mathematical modeling. They studied how a balance is established and controlled between appropriate immune activation and immune response suppression. Segal *et al.* [65] studied the interactions between effector and regulator cells, using it to provide an insight into T cells vaccination. Borghans and de Boer [17] and Borghans *et al.* [18] demonstrated how the interactions of auto-reactive and regulatory T cells can lead to the onset of autoimmunity or stable oscillations around a vaccinated state. León *et al.* [42–44] studied the dynamics of interactions between T cells for the purpose of immune response regulation and control of autoimmune reaction. Blyuss and Nicholson [14, 15] presented a mathematical model for the dynamics of an immune response to a viral infection and autoimmunity, which takes into account T cells with different activation thresholds. They showed how the infection can be cleared by the immune system, as well as showed it could lead to chronic infection or recurrent infection with relapses and remissions, analyzing the potential impact of treatment of autoimmune disease in the chronic and recurrent states.

Burroughs *et al.* [21, 22] examined a mechanism of Treg control of immune responses through regulation of cytokine dependent T cell proliferation, where Tregs affect the level of cytokine through both adsorption and secretion inhibition. They propose that Tregs locally adjust the immune response threshold by inhibiting IL-2 secretion, while assuming that Treg density is controlled through some type of nonlinear competition. Depending on the strength of activation and initial conditions, below a certain threshold of autoimmune antigenic stimulation the autoimmune population may be controlled at low concentrations while Treg population reverts to a homeostatic state. Beyond that threshold the autoimmune population expands and escapes control. At low antigen stimulation levels autoimmune T cells are always controlled for all initial loads. For antigenic stimulation levels between the two thresholds, escape requires the initial load to be sufficiently high. At high antigenic stimulation levels, control is lost and autoimmunity arises. Moreover, even if the antigen stimulation level falls to the original value, at which control was originally achieved, control may not be reacquired, and is only attained if stimulation falls below a second lower threshold. This is a common phenomenon that is termed as hysteresis.

Burroughts *et al.* [23–25] further studied the T cell proliferation thresholds. Burroughs *et al.* [23] determined the analytic formula that describes the fine balance between T cells and Tregs at controlled and immune response equilibrium states. They observed the points where a cusp bifurcation occur and the hysteresis is unfold, showing a drastic change in the dynamical behavior. Burroughs *et al.*

[24] studied how autoimmunity may arise from bystander proliferation of T cells. They observed that if the population of the bystander T cells becomes large enough due to an exposure to pathogen, autoimmunity can arise after a long transient period. In a different article Burroughs *et al.* [25] compared the dynamics of the bystander proliferation between the model with symmetric death rates by Burroughs *et al.* [21] and the model with asymmetric death rates by Pinto *et al.* [57], and observed that the asymmetry allows a slightly faster growth rate of the T cells, in particular for high antigenic stimulations of T cells. Burroughs *et al.* also considered a model with positive correlation between the antigenic stimulation of T cells and the antigenic stimulation of Tregs [23]. They observed the presence of a hysteresis and an isolated region of equilibria, an isola. By increasing the slope parameter, the isola would merge with the hysteresis and at the point of contact we observe a transcritical bifurcation. Oliveira *et al.* [54] further analyzed the model of immune response by $CD4^+$ T cells with the asymmetry presented by Pinto *et al.* [57]. They presented approximate formulas that describe the balance between the concentration of T cells and Tregs for the case not considering a tuning and for the case considering a linear tuning between the antigenic stimulation of T cells and that of Tregs.
In this work, we studied immune responses by $CD4^+$ T cells, with the presence of $CD4^+$ Tregs, using the model initially presented in Burroughs *et al.* [21] with the asymmetry introduced in Pinto *et al.* [57]. In this model, cytokine (e.g. IL-2) dependent growth exhibits a quorum T cell population threshold that determines if immune responses develop on activation [16, 21]. Secretion inhibition by Tregs [67] manipulates the growth dynamics and effectively provides a mechanism for tissue specific regulation of the balance between suppression (control) and immune response. The asymmetry [57] studied in this paper, is modeled by considering that the secreting T cells have a lower death rate than the non secreting T cells and that the active Tregs also have a lower death rate than the inactive Tregs, thus mimicking the effect of the memory T cells. Memory T cells last longer than other T cells and react more promptly to their specific antigen [62]. With this asymmetry, an increase in the antigenic stimulation of T cells results in an increase in the population size of T cells - caused both by the increase in cytokine secretion and by the decrease in the average death rate of T cells, thereby improving the efficiency of the immune responses [23, 24, 57].
The results presented here deepen the results in [21–25, 54, 57]. In particular, we were able to obtain exact formulas for the equilibria, and to compute the eigenvalues to determine the stability. In Section 2.2 we present immune response model as a set of five ordinary differential equations. In Section 2.3 we present the equilibria of the model where we show the explicit formulas that relate the concentration of T cells, the concentration of Tregs, the concentration of interleukine 2 and the antigenic stimulation of T cells. In Section 2.4 we introduce a linear tuning that models a positive correlation between the antigenic stimulation of T cells and the antigenic stimulation of Tregs, and obtain explicit formulas of the equilibria.

2.2 Theory

We consider the immune response model in Section 3 of Burroughs *et al.* [23] and Pinto *et al.* [57], which uses a population of Tregs and conventional T cells with processes shown schematically in Figure 2.1. Both populations of cells require antigenic stimulation for activation. Levels of antigenic

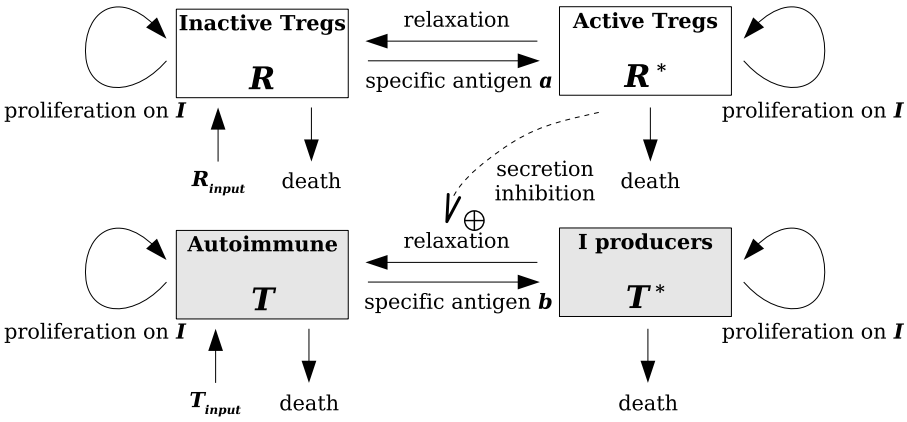

Figure 2.1 Model schematic showing growth, death and phenotype transitions of the Treg populations R, R^*, and autoimmune T cell T, T^* populations. Cytokine dynamics are not shown: IL-2 is secreted by activated T cells T^* and adsorbed by all the T cell populations equally. Reproduced from [23].

stimulation are denoted a and b for Tregs and conventional T cells respectively. Tregs are activated by self antigens from an inactive state, denoted R, to an active state R^*. The IL-2 secreting T cells are denoted T^* and the non secreting T cells are denoted T. On activation, conventional T cells start secreting IL-2 and acquire proliferative capacity in the presence of IL-2. Tregs also proliferate in the presence of IL-2 although less efficiently than normal T cells [70], and they do not secrete IL-2. We include an influx of (auto) immune T cells into the tissue (T_{input}) and Tregs (R_{input}), which can represent T cell circulation or naive T cell input from the thymus. We consider that death may occur independently of other processes or by Fas-FasL induced death [52]. The former terms have equal values for T cells or Tregs but stimulated T cells and Tregs have a lower death rate than relaxed T cells and Tregs. The latter (quadratic) term acts as growth limitation mechanism, assumed to act on all T cells and Tregs equally.

The model consists of a set of ordinary differential equations. We have five compartments: one for each T cell population - inactive Tregs R, active Tregs R^*, non secreting T cells T, secreting activated T cells T^*, and one for interleukin 2 density I.

Parameter	Symbol	Range	Value
T cell T, T^*			
T cell Maximum growth rate[1]	ρ/α	< 6 day^{-1}	4 day^{-1}
Death rate of inactive T cells (day^{-1})	d_T	$0.1 - 0.01$ [50]	0.1
Death rate ratio of active : inactive T cells	d_{T^*}/d_T	$0.01 - 100$	0.1
Capacity of T cells[2]	$\rho/(\alpha\beta)$	$10^6 - 10^7$ cells/ml [51]	10^7 cells/ml
Input rate of inactive T cells (cells/ml/day)	T_{input}	$0 - 10^4$	100
Secretion reversion (constant)[3]	k	hrs-days	0.1 hr^{-1}
Antigen stimulation level	bk	0.001-200 $\times a\hat{k}$	Bifurcation parameter
Tregs R, R^*			
Growth rate ratio T_{reg}:T	ε	< 1	0.6
Homeostatic capacity R_{hom}	$(\varepsilon\rho S/\hat{d} - \hat{\delta})/\hat{\alpha}$	$10 - 10^5$ cells/ml	10^4 cells/ml
Relaxation rate	$\hat{k}$	hrs-days	0.1 hr^{-1}
Death rate ratio of inactive Tregs : inactive T cells	d_R/d_T	$0.01 - 100$	1
Death rate relative ratio of Tregs : T cells	$\frac{d_{R^*}}{d_R}/\frac{d_{T^*}}{d_T}$	$0.01 - 100$	1
Input rate ratio of inactive Tregs : inactive T cells	R_{input}/T_{input}	$0 - 10^2$	1
T_{reg} antigen stimulation level	$a\hat{k}$	0-10 per day	1 per day
Secretion inhibition[4]	γ	0.1-100 $\times R_{hom}^{-1}$	10 R_{hom}^{-1}
Cytokines			
Max. cytokine concentration[5]	$1/\alpha$	100-500 pM	200 pM
IL2 secretion rate	σ	0.07, 2 fgrms h^{-1} [71][6]	10^6 molecs s^{-1} cell^{-1}
Cytokine decay rate	$\sigma\delta$	hrs-days	1.5 hr $^{-1}$ [2]

Table 2.1 Parameters values for our model of T cells and Tregs, adopted from [21] and [54].

[1]Minimum duration of SG_2M phase $\alpha\rho^{-1} \approx 3$ hrs.
[2]Maximum T cell density for severe infections (based on $LCMV$).
[3]This is in absence of Tregs.
[4]This is in terms of the homeostatic Treg level R_{hom} which is set to 10^4 cells per ml.
[5]This is taken as 20 times the receptor affinity ($10pM$).
[6]Naive and memory cells respectively. This corresponds to $3 \times 10^3 - 10^5$ molecules per h, IL2 mass 15–18 kDa.

$$\begin{aligned}
\frac{dR}{dt} &= (\varepsilon\rho I-\beta(R+R^*+T+T^*)-d_R)R+\hat{k}(R^*-aR)+R_{input},\\
\frac{dR^*}{dt} &= (\varepsilon\rho I-\beta(R+R^*+T+T^*)-d_{R^*})R^*-\hat{k}(R^*-aR),\\
\frac{dT}{dt} &= (\rho I-\beta(R+R^*+T+T^*)-d_T)T+k(T^*-bT+\gamma R^*T^*)+T_{input},\\
\frac{dT^*}{dt} &= (\rho I-\beta(R+R^*+T+T^*)-d_{T^*})T^*-k(T^*-bT+\gamma R^*T^*),\\
\frac{dI}{dt} &= \sigma(T^*-(\alpha(R+R^*+T+T^*)+\delta)I).
\end{aligned}$$

The parameters of our model and their default values are presented in Table 2.1

2.3 Equilibria of the model

In an ODE model, the equilibria, stable or unstable, is the set of points where all the derivatives vanish. Let $x = T+T^*$ be the total concentration of T cells and let $y = R+R^*$ be the total concentration of Tregs. When the system is at equilibrium we have that:

$$(\varepsilon\rho I-\beta(x+y)-d_R)R+\hat{k}(R^*-aR)+R_{input}=0\,, \tag{2.1}$$
$$(\varepsilon\rho I-\beta(x+y)-d_{R^*})R^*-\hat{k}(R^*-aR)=0\,, \tag{2.2}$$
$$(\rho I-\beta(x+y)-d_T)T+k(T^*-bT+\gamma R^*T^*)+T_{input}=0\,, \tag{2.3}$$
$$(\rho I-\beta(x+y)-d_{T^*})T^*-k(T^*-bT+\gamma R^*T^*)=0\,, \tag{2.4}$$
$$\sigma(T^*-(\alpha(x+y)+\delta)I)=0\,. \tag{2.5}$$

In Theorem 1 we present rational fractions that, given the concentration of T cells x and the concentration of Tregs y, allow us to compute the interleukine 2 density I, the concentration of secreting T cells T^*, the concentration of active Tregs R^* and the *antigen function* $b(x,y)$ that relates the concentration of T cells x and the concentration of Tregs y with the level of the antigenic stimulation of T cells b. Moreover, we have obtained a polynomial that gives the exact *balance* between the concentration of T cells x and the concentration of Tregs y (see Figures 2.2 and 2.3). Let A and B be such that $A(x,y)=\alpha(x+y)+\delta$ and $B(x,y)=\beta(x+y)$. Let U, L and M be such that

$$\begin{aligned}
U(x,y) &= (B+d_T)x-T_{input},\\
L(x,y) &= \rho x+(d_T-d_{T*})A,\\
M(x,y) &= \left(\left(B+d_{R^*}+\hat{k}(1+a)\right)L-\varepsilon\rho U\right)/\hat{k}a.
\end{aligned}$$

Theorem 1. *At equilibrium I, T^* and R^* are given by*

$$I(x,y)=\frac{U}{L},\quad T^*(x,y)=AI,\quad R^*(x,y)=\frac{yL}{M},$$

and the antigen function b is given by

$$b(x,y)=(\beta(x+y)+d_{T^*}+k(1+\gamma R^*)-\rho I)T^*/\left(k(x-T^*)\right)\,.$$

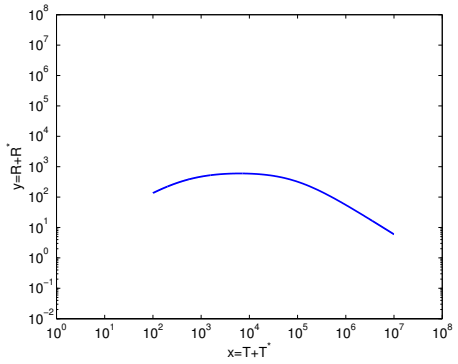

Figure 2.2 Balance between the concentration of T cells $x = T + T^*$ and the concentration of Tregs $y = R + R^*$. The figure represents a cross section of the equilibria manifold obtained from Theorem 1 for the default values of the parameters.

Furthermore, the balance *between the concentration of T cells and Tregs is given by*

$$(\varepsilon\rho U y + L(R_{input} - y(B + d_R)))\, M + (d_R - d_{R*}) y L^2 = 0\ .$$

We note that the above values of I, T^* and R^* do not depend upon b and are rational functions on x and y.

Proof. By Equation (2.5), using the definition of A

$$T^* = AI\ .$$

In order to eliminate the terms with b, we add Equations (2.3) and (2.4), obtaining

$$(\rho I - B)(T + T^*) - d_T T - d_{T^*} T^* + T_{input} = 0\ .$$

Noting that $x = T + T^*$, we get

$$(\rho T^*/A - B)x - d_T(x - T^*) - d_{T^*} T^* + T_{input} = 0\ .$$

Hence,

$$T^* = \frac{((B + d_T)x - T_{input})A}{\rho x + (d_T - d_{T*})A}\ .$$

This proves the formulas for I and T^*.
Let us prove the formula for R^*. Applying $y = R + R^*$ in Equation (2.2), we get

$$(\varepsilon\rho U/L - (B + d_{R^*}))\, R^* - \hat{k}\,(R^* - a(y - R^*)) = 0\ .$$

Hence,

$$R^* = \frac{a\hat{k} L y}{\left(B + d_{R^*} + \hat{k}(1 + a)\right) L - \varepsilon\rho U}\ .$$

This proves the formula for R^*. By Equation (2.4), noting that $x = T + T^*$, we have

$$(\rho I - \beta(x+y) - d_{T^*})T^* - k(T^* - b(x-T^*) + \gamma R^* T^*) = 0 \, .$$

Hence,

$$b = (\beta(x+y) + d_{T^*} + k(1+\gamma R^*) - \rho I)T^* / \left(k(x-T^*)\right) \, .$$

This proves the formula for b. Let us prove the balance formula. Adding Equations (2.1) and (2.2), we obtain

$$(\varepsilon\rho I - \beta(x+y))y - (d_R R + d_{R^*} R^*) + R_{input} = 0 \, .$$

Hence,

$$(\varepsilon\rho U/L - B)y - (d_R y - (d_R - d_{R*})R^*) + R_{input} = 0 \, .$$

□

We computed numerically the eigenvalues (see Figure 2.3) using the Jacobian of the ODE system, in terms of the pair (x,y), that is given by the following matrix:

$$J(x,y) = f(R(x,y), R^*(x,y), T(x,y), T^*(x,y), I(x,y))$$
$$= \begin{pmatrix} J_{11} & -\beta R + \hat{k} & -\beta R & -\beta R & \varepsilon\rho R \\ -\beta R^* + \hat{k}a & J_{22} & -\beta R^* & -\beta R^* & \varepsilon\rho R^* \\ -\beta T & -\beta T + k\gamma T^* & J_{33} & -\beta T + k + k\gamma R^* & \rho T \\ -\beta T^* & -\beta T^* - k\gamma T^* & -\beta T^* + kb & J_{44} & \rho T^* \\ -\sigma\alpha I & -\sigma\alpha I & -\sigma\alpha I & \sigma - \sigma\alpha I & J_{55} \end{pmatrix}$$

where

$$\begin{aligned} J_{11} &= \varepsilon\rho I - \beta(R+x+y) - d_R - \hat{k}a, \\ J_{22} &= \varepsilon\rho I - \beta(R^*+x+y) - d_{R^*} - \hat{k}, \\ J_{33} &= \rho I - \beta(T+x+y) - d_T - kb, \\ J_{44} &= \rho I - \beta(T^*+x+y) - d_{T^*} - k - k\gamma R^*, \\ J_{55} &= -\sigma(\alpha(x+y) + \delta) \, . \end{aligned}$$

We observe that for the parameters considered, using the balance equation, we have that the concentration of Tregs y is also a function of the concentration of T cells x (see Figure 2.2). Hence, the stability of the equilibria and the bifurcation boundary can be characterized only in terms of the concentration of T cells x. By Theorem 1, all the equilibria points are characterized in terms of the pairs (x,y) satisfying the balance equation. Hence, their stability (or instability) is also characterized in terms of the pairs (x,y) satisfying the balance equation. The bifurcation boundary $\mathscr{B}$ is the set of equilibria points (R, R^*, T, T^*, I) with the property that at least one of the eigenvalues has real part equal to zero and all the other eigenvalues have non positive real part. Therefore, using Theorem 1, the bifurcation boundary $\mathscr{B}$ can be fully characterized in terms of the pairs (x,y) satisfying the balance equation. By Theorem 1, the antigenic stimulation of T cells (parameter b) is fully characterized by the pair (x,y) satisfying the balance equation. Hence, the projection of the bifurcation boundary $\mathscr{B}$ in the antigenic stimulation of T cells, parameter b, is well characterized, resulting in a lower threshold b_L of antigenic stimulation of T cells and a higher threshold b_H of antigenic stimulation of T cells

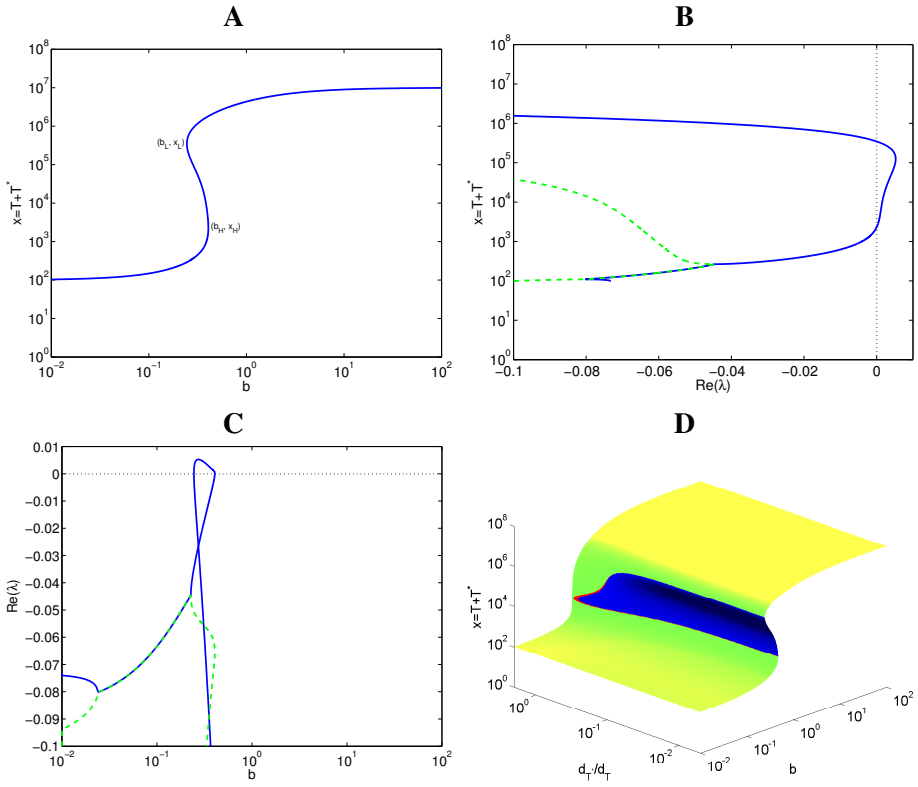

Figure 2.3 **A**: Antigen function $b(x,y)$ that relates the antigenic stimulation b of T cells, the concentration of T cells $x = T + T^*$ and the concentration of Tregs $y = R + R^*$. The points (b_L, x_L) and (b_H, x_H) indicate the saddle-node bifurcation points where the stability changes.
B and **C**: Relation between the eigenvalue with the largest real part (blue line) and the eigenvalue with the second largest real part (green dashes) with the concentration x of T cells (**B**) and with the antigenic stimulation b of T cells (**C**). For the presented values of the parameters, we observe that the first eigenvalue is positive for x in $(x_H, x_L) \approx (2.31 \times 10^3, 3.47 \times 10^5)$ and b in $(b_L, b_H) \approx (0.245, 0.407)$; and that the second eigenvalue is negative.
D: Relation between antigenic stimulation b of T cells and the concentration of T cells x, when changing the parameter d_{T^*}/d_T between 0.005 and 2, using Theorem 1. Horizontal axis: b; "away axis": d_{T^*}/d_T; vertical axis: $x = T + T^*$. The colors indicate the stability of the equilibria: yellow to green for negative eigenvalues; red for eigenvalues close to zero; and blue to black for positive eigenvalues.

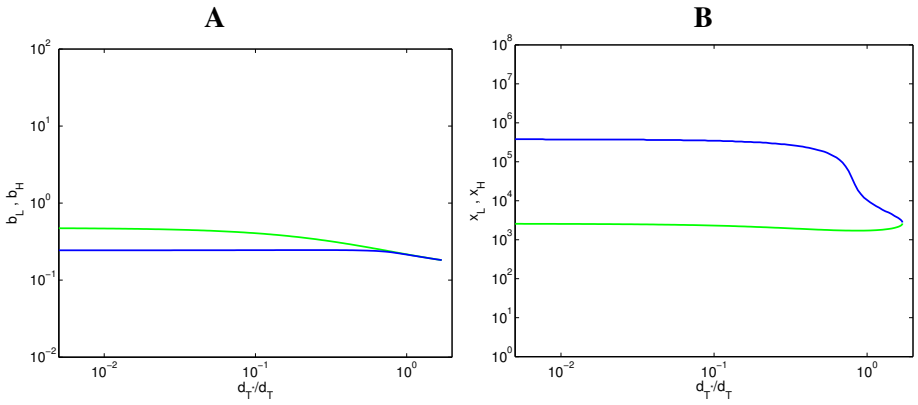

Figure 2.4 Unfolding of the hysteresis (see also Figure 2.3D). We observe a cusp bifurcation for $d_{T^*}/d_T \approx 1.71$. Inside the bounded region we found 2 stable equilibria and 1 unstable equilibria; outside of the bounded region we found 1 stable equilibria.
A: Thresholds of antigenic stimulation of T cells, b_L (blue) and b_H (green).
B: Thresholds of the concentration of T cells, x_L (blue) and x_H (green).

(see Figures 2.3A and 2.4). For an antigenic stimulation of T cells below the threshold b_L, we observe one stable equilibrium, a controlled state, characterized by a low concentration of T cells. For an antigenic stimulation of T cells above the threshold b_H we observe a stable equilibrium, an immune response state, characterized by a high concentration of T cells. Between the two antigenic thresholds, b_L and b_H we observe two stable equilibria, a controlled state and an immune response state, and for intermediate concentrations of T cells we observe one unstable equilibrium, that belongs to the separatrix of the basins of the attraction of the stable equilibria. For the default values of the other parameters, we observe the unfolding of the hysteresis when we increase the parameter d_{T^*}/d_T from 0.1 up to ≈ 1.71 where a cusp bifurcation occurs.

2.4 The effect of a linear tuning between the antigenic stimulations of T cells and Tregs

Here, we consider the same ordinary differencial Equations in Section 2.2 with the addition of a linear tuning $\hat{a} = a + mb$, as in Burroughs *et al.* [23] and Pinto *et al.* [57], to model a positive correlation between the antigenic stimulation b of T cells, and the antigenic stimulation $\hat{a}$ of Tregs. i.e.

$$\begin{aligned}
\frac{dR}{dt} &= (\varepsilon\rho I - \beta(R+R^*+T+T^*) - d_R)R + \hat{k}(R^* - aR - mbR) + R_{input}, \\
\frac{dR^*}{dt} &= (\varepsilon\rho I - \beta(R+R^*+T+T^*) - d_{R^*})R^* - \hat{k}(R^* - aR - mbR), \\
\frac{dT}{dt} &= (\rho I - \beta(R+R^*+T+T^*) - d_T)T + k(T^* - bT + \gamma R^* T^*) + T_{input}, \\
\frac{dT^*}{dt} &= (\rho I - \beta(R+R^*+T+T^*) - d_{T^*})T^* - k(T^* - bT + \gamma R^* T^*), \\
\frac{dI}{dt} &= \sigma(T^* - (\alpha(R+R^*+T+T^*) + \delta)I).
\end{aligned}$$

The equilibria is the set of points such that:

$$(\varepsilon\rho I - \beta(x+y) - d_R)R + \hat{k}(R^* - aR - mbR) + R_{input} = 0\,, \tag{2.6}$$
$$(\varepsilon\rho I - \beta(x+y) - d_{R^*})R^* - \hat{k}(R^* - aR - mbR) = 0\,, \tag{2.7}$$
$$(\rho I - \beta(x+y) - d_T)T + k(T^* - bT + \gamma R^* T^*) + T_{input} = 0\,, \tag{2.8}$$
$$(\rho I - \beta(x+y) - d_{T^*})T^* - k(T^* - bT + \gamma R^* T^*) = 0\,, \tag{2.9}$$
$$\sigma(T^* - (\alpha(x+y) + \delta)I) = 0\,. \tag{2.10}$$

The parameter $a = 1/2.4$ per day measures the background antigenic stimulation of Tregs when $b = 0$, and m is the slope parameter, chosen to be in $[0,1]$. We present here explicit formulas for the equilibria, stable or unstable, that relate the concentration of T cells x, the concentration of Tregs y and the antigenic stimulation of T cells b.

Let A, B, U, L, W, C, E, F, G and H be such that

$$\begin{aligned}
A(x,y) &= \alpha(x+y) + \delta \\
B(x,y) &= \beta(x+y) \\
U(x,y) &= (B + d_T)x - T_{input} \\
L(x,y) &= \rho x + (d_T - d_{T^*})A \\
W(x,y) &= \varepsilon\rho U - BL \\
C(x,y) &= d_R yL - Wy - R_{input}L \\
E(x,y) &= (((B + d_{T^*} + k)L - \rho U)(d_R - d_{R^*}) + k\gamma C)AU \\
F(x,y) &= k(xL - AU)(d_R - d_{R^*})L \\
G(x,y) &= W - (d_{R^*} + \hat{k})L \\
H(x,y) &= \hat{k}((d_R - d_{R^*})Ly - C)L
\end{aligned}$$

We note that the functions A, B, U and L are the same as the ones presented in Section 2.3.

Theorem 2. *At equilibrium I, T^* and R^* are given by*

$$I(x,y) = \frac{U}{L}, \quad T^*(x,y) = AI, \quad R^*(x,y) = \frac{C}{(d_R - d_{R^*})L}$$

and the antigen function b is given by

$$b(x,y) = \frac{E}{F} .$$

Furthermore, the balance between the concentration of T cells and Tregs is given by

$$CFG + (aF + mE)H = 0 .$$

Proof. By Equation (2.10), using the definition of A

$$T^* = AI .$$

Adding Equations (2.8) and (2.9) we obtain

$$(\rho I - B)(T + T^*) - d_T T - d_{T^*} T^* + T_{input} = 0 .$$

Noting that $x = T + T^*$, we get

$$(\rho T^*/A - B)x - d_T(x - T^*) - d_{T^*} T^* + T_{input} = 0 .$$

Hence,

$$T^* = \frac{((B + d_T)x - T_{input})A}{\rho x + (d_T - d_{T*})A} .$$

This proves the formulas for I and T^*.
To prove the formula for R^*, we add Equations (2.6) and (2.7), obtaining

$$(\varepsilon\rho I - \beta(x+y))\, y - d_R R - d_{R^*} R^* + R_{input} = 0 .$$

Nothing that $y = R + R^*$ and using the definition of B and I we get

$$\left(\varepsilon\rho \frac{U}{L} - B\right) y - d_R y + (d_R - d_{R^*})R^* + R_{input} = 0 .$$

Multiplying by L and using the definition of C we have

$$-C + (d_R - d_{R^*})LR^* = 0 .$$

Now, let us prove the formula for b. From Equation (2.9) and noting that $x = T + T^*$ we obtain

$$b = \frac{((B + d_{T^*} + k - \rho I) + k\gamma R^*)T^*}{k(x - T^*)} .$$

Using the expressions for I and T^*, and multiplying the numerator and denominator by L

$$b = \frac{(((B+d_{T^*}+k)L-\rho U)+k\gamma LR^*)AU}{k(xL-AU)L} .$$

Using the equation for R^* and multiplying the numerator and the denominator by $(d_R - d_{R^*})$

$$b = \frac{(((B+d_{T^*}+k)L-\rho U)(d_R-d_{R^*})+k\gamma C)AU}{k(xL-AU)(d_R-d_{R^*})L} .$$

Let us prove the balance equation between x and y. Applying $y = R+R^*$ and the definition of B and I in Equation (2.7), we get

$$(\varepsilon\rho U/L - B - d_{R^*})R^* - \hat{k}R^* + \hat{k}a(y-R^*) + \hat{k}mb(y-R^*) = 0 .$$

Multiplying by L

$$\left(\varepsilon\rho U - BL - (d_{R^*}+\hat{k})L\right)R^* + \hat{k}\left(a+m\frac{E}{F}\right)(y-R^*)L = 0 .$$

Using the expressions for G and R^*

$$\frac{CG}{(d_R-d_{R^*})L} + \hat{k}\left(\frac{aF+mE}{F}\right)\left(y - \frac{C}{(d_R-d_{R^*})L}\right)L = 0 .$$

We finish the proof by dividing the above equation by $(d_R - d_{R^*})LF$ and using the definition of H.

□

After obtaining the equilibria, we can assess the stability by computing numerically the eigenvalues using the Jacobian of the ODE system given by

$$J(x,y) = f(R(x,y),R^*(x,y),T(x,y),T^*(x,y),I(x,y))$$
$$= \begin{pmatrix} J_{11} & -\beta R+\hat{k} & -\beta R & -\beta R & \varepsilon\rho R \\ -\beta R^*+\hat{k}(a+mb) & J_{22} & -\beta R^* & -\beta R^* & \varepsilon\rho R^* \\ -\beta T & -\beta T+k\gamma T^* & J_{33} & -\beta T+k+k\gamma R^* & \rho T \\ -\beta T^* & -\beta T^*-k\gamma T^* & -\beta T^*+kb & J_{44} & \rho T^* \\ -\sigma\alpha I & -\sigma\alpha I & -\sigma\alpha I & \sigma-\sigma\alpha I & J_{55} \end{pmatrix}$$

where

$$\begin{aligned} J_{11} &= \varepsilon\rho I - \beta(R+x+y) - d_R - \hat{k}(a+mb), \\ J_{22} &= \varepsilon\rho I - \beta(R^*+x+y) - d_{R^*} - \hat{k}, \\ J_{33} &= \rho I - \beta(T+x+y) - d_T - kb, \\ J_{44} &= \rho I - \beta(T^*+x+y) - d_{T^*} - k - k\gamma R^*, \\ J_{55} &= -\sigma(\alpha(x+y)+\delta) . \end{aligned}$$

We observe that the balance between the concentration of T cells and the concentration of Tregs varies with the slope parameter m. For low values of the slope parameter m, we obtain an hysteresis with its bistability region. As we further increase the slope parameter m, we find up to three possible values of

the concentration of the Tregs for each value of the concentration of the T cells.
When we increase the slope parameter from 0 to 1, we observe that the hysteresis changes slightly in shape but not in the number of equilibria. At $m = 5.7 \times 10^{-3}$ we observe the appearance of a single point, an isola, a saddle-node equilibria for $b = 1.3 \times 10^2 > b_H$ and $x = 1.0 \times 10^3$.
As we further increase the slope m, we observe that the isola increases. The isola presents a stable equilibria for lower concentrations of T cells and an unstable equilibria for higher concentration of T cells. It eventually reaches the hysteresis for $m = 4.1 \times 10^{-2}$ creating a transcritical bifurcation at $b = 1.0$ and $x = 1.0 \times 10^3$. For larger values of the slope m we observe a hysteresis considerably wider than the one obtained for m close to zero.

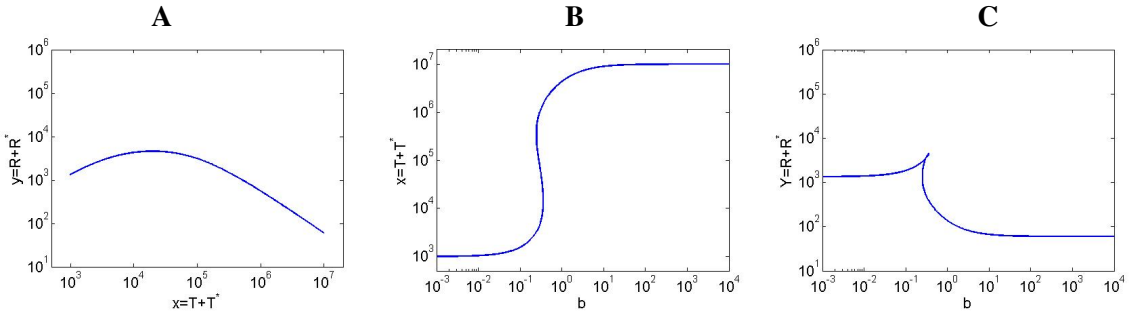

Figure 2.5 Cross section of the equilibria manifold obtained from Theorem 2 for the default values of the parameters and with the slope parameter $m = 0$, equal to the model without tuning. **A**: Balance between the concentration of T cells $x = T + T^*$ and the concentration of Tregs $y = R + R^*$. **B**: Relationship between the antigenic stimulation b of T cells and the concentration of T cells $x = T + T^*$. **C**: Relationship between the antigenic stimulation b of T cells and the concentration of Tregs $y = R + R^*$.

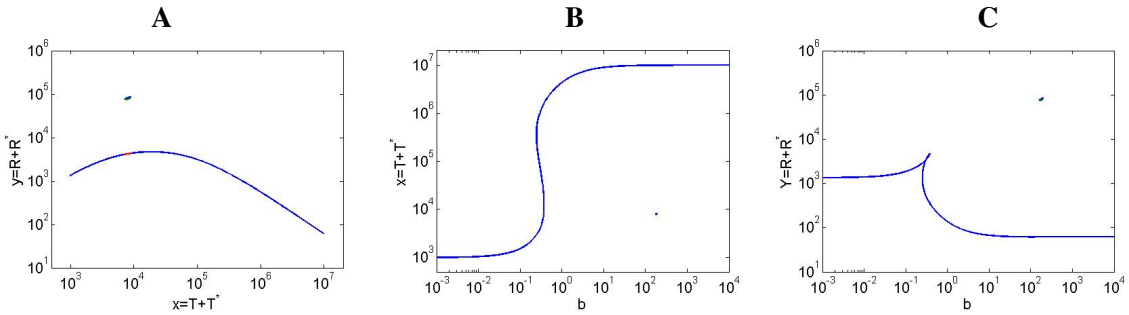

Figure 2.6 Cross section of the equilibria manifold obtained from Theorem 2 for the default values of the parameters and with the slope parameter $m \approx 5.7 \times 10^{-3}$, appearance of the isola. **A**: Balance between the concentration of T cells $x = T + T^*$ and the concentration of Tregs $y = R + R^*$. **B**: Relationship between the antigenic stimulation b of T cells and the concentration of T cells $x = T + T^*$. **C**: Relationship between the antigenic stimulation b of T cells and the concentration of Tregs $y = R + R^*$.

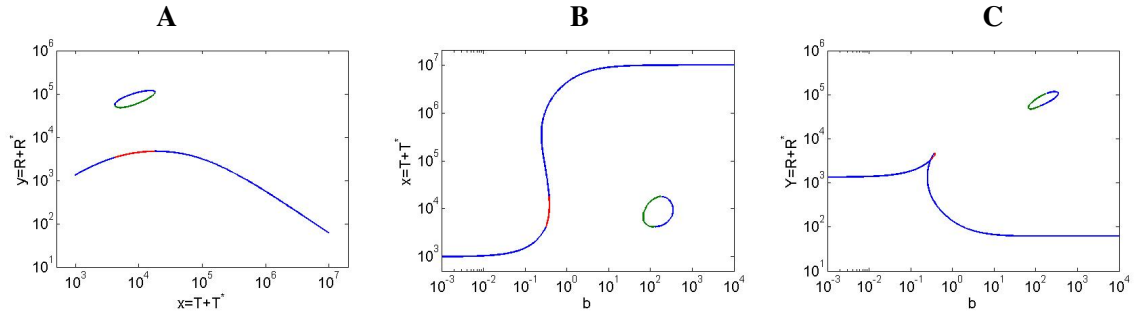

Figure 2.7 Cross section of the equilibria manifold obtained from Theorem 2 for the default values of the parameters and with the slope parameter $m = 7.0 \times 10^{-3}$, appearance of a loop from the isola. **A**: Balance between the concentration of T cells $x = T + T^*$ and the concentration of Tregs $y = R + R^*$. **B**: Relationship between the antigenic stimulation b of T cells and the concentration of T cells $x = T + T^*$. **C**: Relationship between the antigenic stimulation b of T cells and the concentration of Tregs $y = R + R^*$.

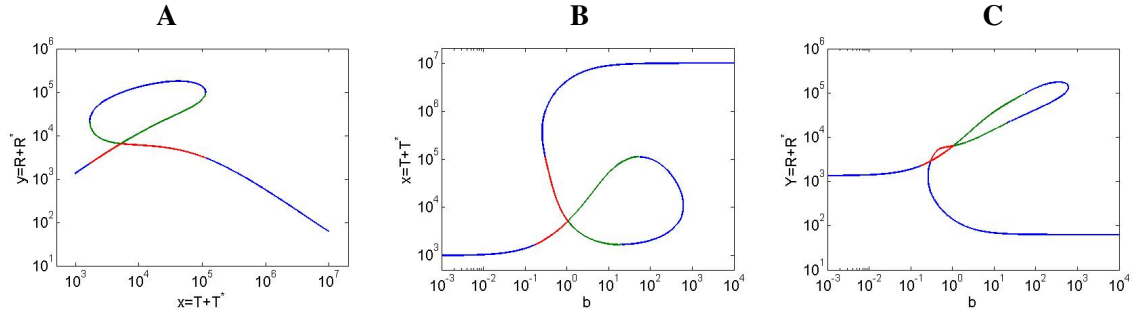

Figure 2.8 Cross section of the equilibria manifold obtained from Theorem 2 for the default values of the parameters and with the slope parameter m around 4.1×10^{-2}, a transcritical bifurcation occurs at $b \approx 1, x \approx 6 \times 10^3, y \approx 6 \times 10^3$. **A**: Balance between the concentration of T cells $x = T + T^*$ and the concentration of Tregs $y = R + R^*$. **B**: Relationship between the antigenic stimulation b of T cells and the concentration of T cells $x = T + T^*$. **C**: Relationship between the antigenic stimulation b of T cells and the concentration of Tregs $y = R + R^*$.

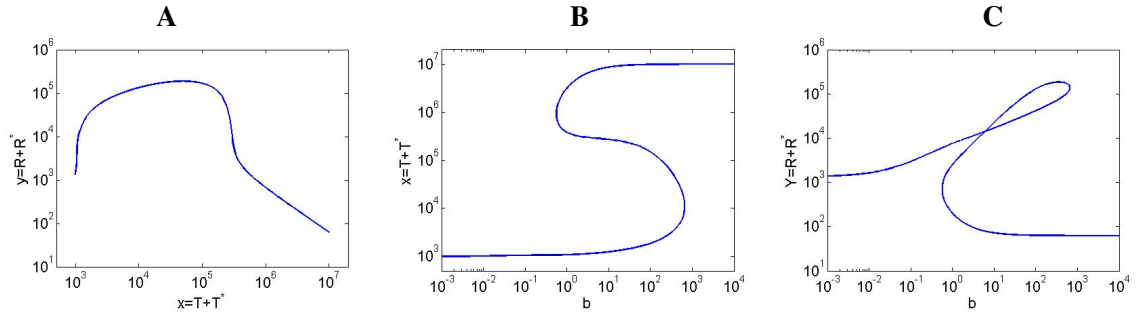

Figure 2.9 Cross section of the equilibria manifold obtained from Theorem 2 for the default values of the parameters and with the slope parameter $m = 1$, with a considerably wider hysteresis. **A**: Balance between the concentration of T cells $x = T + T^*$ and the concentration of Tregs $y = R + R^*$. **B**: Relationship between the antigenic stimulation b of T cells and the concentration of T cells $x = T + T^*$. **C**: Relationship between the antigenic stimulation b of T cells and the concentration of Tregs $y = R + R^*$.

Chapter 3

Convergence to Walrasian prices in random matching Edgeworthian economies with bounded rationality

3.1 Introduction

General equilibrium analysis is widely used in modern economics, it gives us insights as to what we can expect, in economic terms, from the complex interaction between participants, markets, and institutions, in a social medium. It was brought to attention following the work of Walras in the 1870s who first stated the theory of exchange in its generality [74], and the work of Edgeworth [31] who stated the theory of market behavior, and was the first to study the core, under the name 'contract curve'. These interactions have influence not only on the individual markets but also on the aggregate economy. The theorems of welfare economics by Arrow [3] imply that the distortion of relative commodity prices will have a social cost when analyzed using a general equilibrium model, since policies such as taxes and tariffs could move the economy away from a Pareto optimal allocation, with costly effects.

General equilibrium analysis attempts to explain the behavior of markets: behavior of demand, supply, and prices, by seeking to prove that the interaction of demand and supply will result in a general equilibrium. Arrow and Debreu [5] suggested that under certain economic assumptions such as convex preferences, independence of demand and perfect competition there must be a set of prices such that aggregate supply equals aggregate demand for every commodity in the economy.

The Walrasian general equilibrium model assumes that consumers are passive price takers - they regard a given set of prices as parameters in determining their optimal net demands and supplies. The equilibrium price is such that the market clears, then the consumers change their endowments by the allocations determined by the equilibrium price. A mechanism that leads to the equilibrium price can be achieved, for instance, through an auctioneer who collects all the offers and demands for each good and adjusts the price vector to clear the market. Here, each consumer has an initial endowment of the commodities and a set of preferences, resulting in demand functions for each commodity, where market demand is the sum of consumers' demands. The Commodity market demands depend on all

prices, and are continuous, nonnegative, homogeneous of degree zero, and satisfy Walras's law (i.e. at any set of prices, the total value of consumer expenditures equals consumer incomes). Consumers maximize utility subject to their budget constraint, producers (in models with production) maximize profits. In equilibrium, market prices are such that the required equilibrium conditions hold; demand equals supply for all commodities, and on the production side (for models with production) zero profit conditions are satisfied for each industry. On the other hand, the Edgeworthian concept considers consumers as active market participants trading with each other in an attempt to reach a higher level of utility. According to this point of view, an equilibrium is achieved when no person participating in the market can become better off without another person becoming worse off.

The study of market games has been an outstanding problem in the field of economics, and mathematical economics in particular. One of the objectives is to provide a market game which can approximately enough describe the behavior of participants and of markets in real situations such that the equilibrium of this approximate market approaches the competitive equilibrium of the same market. A popular approach is the use of random matching games where participants meet randomly and exchange goods according to some set rules. Many researchers developed the work in these directions, among them Binmore and Herrero [11], Shaked and Sutton [66], Rubinstein and Wolinsky [61], MasColell [47], McLennan and Schonnesschein [49], Wooders [75], Lloyd [46], Dubey and Geanokoplos [29], Harves-Beloso *et al.* [39], Levin [45], Voorneveld [73], and Gale [33–36].

In order to isolate the effect of market rules from the participants' behavior, Gode and Sunder [37, 38] introduced 'zero intelligence' programs in place of human traders, where programs submit offers and bids randomly, subject to a budget constraint. They showed that a double auction, a non-Walrasian market mechanism, can sustain high level of allocative efficiency even if participants do not maximize profit, suggesting that allocative efficiency of a double auction is derived largely from its structure (market rules imposed on participants), independent of trader's motivation, intelligence or learning.

With the help of Gale's results [36], Pinto *et al.* [58] showed that for a class of random matching Edgeworth economies with two goods and participants, with preferences, described by Cobb-Douglas utility, the expectation of the limiting equilibrium price coincides with the equilibrium price of the related Walrasian economies. Furthermore, Pinto *et al.* [58] extended the results to study economies in the presence of uncertainty within the framework of multi-period Arrow-Debreu model in order to show how beliefs survive and propagate through markets, providing a framework on which to study markets in which there is no enough data that will enable detailed study to predict the probability of future states. Ferreira *et al.* [32] extended the model of Pinto *et al.* [58] by associating to each participant a bargaining skill (or a selfishness) factor, allowing trade to occur to a point in the core different from the bilateral equilibrium. In this work, we extended the work in Pinto et al [58], and Ferreira et al [32] by allowing trade to occur outside the core, inspired by the 'zero intelligence' participants of Gode and Sunder [37, 38]. We studied conditions under which the equilibrium of the market, defined by a random matching game as in Binmore and Herrero [12], approaches the equilibrium of a fully competitive Walrasian model. The chapter is divided into four sections, section 3.2 presents a description of the general equilibrium model, section 3.3 explains the duality concepts, and section 3.4 has our main result.

3.2 Economic model

We look at a dynamic matching and bargaining game $G = \{(Z_i, e_i, u_i)\}_{i=1}^{N}$, where each participant has a consumption set $Z_i \subset \mathbb{R}_+^M$, a consumption bundle e_i and a utility function u_i. We will consider the case of two goods being traded, i.e. $M = 2$, $z_i = (x_i, y_i) \in \mathbb{R}_+^2$ and utility function of the Cobb-Douglas type:

$$u_i(x_i, y_i) = x_i^{\alpha_i} y_i^{1-\alpha_i}; \quad \text{with preference } \alpha_i \ . \tag{3.1}$$

Let $\mathscr{A} = \{A_1, A_2, ..., A_N\}$ be a collection of N participants. Participants A_i have preferences α_i. Some given amounts of goods, each with positive price $p > 0$ are distributed among the N participants, such that participant A_i owns an initial endowment (x_i, y_i) of goods x and y. The marginal rate of substitution MRS of participant A_i is given by

$$MRS(A_i) = \frac{\frac{\partial u_i(A_i)}{\partial x_i}}{\frac{\partial u_i(A_i)}{\partial y_i}} \ .$$

From Equation 3.1 we obtain

$$\frac{\partial u_i}{\partial x_i} = \frac{\alpha_i}{x_i} u_i \ ,$$
$$\frac{\partial u_i}{\partial y_i} = \frac{(1-\alpha_i)}{y_i} u_i \ .$$

Hence,

$$MRS(A_i) = \frac{\frac{\alpha_i}{x_i}}{\frac{1-\alpha_i}{y_i}} = \frac{\alpha_i y_i}{(1-\alpha_i)\, x_i} \ .$$

Since participants have different endowments and preferences, then there may be gains from trading. Thus, participants may exchange commodities in order to maximize their preferences. Ideally, given a price vector p, each participant would choose a consumption to maximize her utility given her budget constraint. Therefore, participant A_i would solve the problem

$$max_{z \in \mathbb{R}_+^2} u_i(z) \text{ such that } p.z \leq p.e_i \ . \tag{3.2}$$

3.2.1 The Walrasian Equilibrium

A Walrasian equilibrium is a vector of prices, and a consumption bundle for each agent, such that every agent's consumption maximizes her utility given prices, and the market clears: the total demand for each commodity equals the aggregate endowment.

Definition 1. *A **Walrasian equilibrium** for the economy G is a vector consisting of the price and final allocations (p, z_i) such that:*

1.

$$\textit{For all } i, \ z_i \in \underset{p.z_i \leq p.e_i}{\arg\max}\, u_i(z)$$

2.

$$\textit{For all } i,\ \sum_i z_i = \sum_i e_i$$

Therefore, a competitive equilibrium consists of a price structure p at which the total supply of each commodity balances the total demand, and the allocation z that results from trading at these prices is the equilibrium allocation. I.e. z here is an allocation with the property that at the price structure p, no participant can, within the value of his initial bundle, buy a bundle that he prefers to z. If no coalition of participants can enforce an outcome that is better for them than z, then z is said to be in the core of the market. Hence, the core is the set of Pareto-optimal points not blocked by any participant. The notion of competitive equlibria holds only when there is perfect competition, otherwise the decision of one participant can change prevailing prices such that the restrictions on the prices will not hold. On the other hand, the notion of the core does not depend on perfect competition, it can hold even for markets consisting of two or three participants. It involves only direct trading between participants, against the notion of competitive equilibrium which assumes that traders allow market pressure to determine prices, and then trade passively according to these prices. Even though every equilibrium allocation is always in the core, the core of a finite market usually contains points that are not equilibrium allocations. Aumann [8] showed that the core approaches the set of equilibrium allocations when the number of participants approaches infinity.
When participants follow Cobb-Douglas utility functions, the Walrasian equilibrium price p_W determines the unique point in the core such that the market "locally" clears:

$$p_W(\mathscr{A}) = \frac{\sum_{i=1}^{N} \alpha_i y_i}{\sum_{i=1}^{N} (1-\alpha_i) x_i}\ . \tag{3.3}$$

3.2.2 The bilateral equilibrium

Bilateral trade is the trade between two participants A_i and A_j. The case with two commodities, x and y, is the well known scenario analyzed in the Edgeworth box diagram represented in Figure 3.1.
We represent in the Edgeworth box, the indifference curves for both participants, i.e. the curves of constant utility, $u_i = constant$ and $u_j = constant$, passing through the point corresponding to the initial endowments of both participants. Since we are using Cobb-Douglas utility function, the indifference curve for participant A_i has the formula

$$y = y_i \left(\frac{x}{x_i}\right)^{\frac{\alpha_i}{1-\alpha_i}}\ .$$

The contract curve is the curve where the indifference curves of both participants are tangent and such that the utilities of both participants are greater than or equal to the initial ones. It is the set of all Pareto-optimal allocations, where the condition for Pareto-optimality is that both participants have equal values of the marginal rate of substitution. Hence, at the contract curve, neither participant can improve her utility without worsening the utility of the other participant. In our model, the contract curve is

$$y = \frac{(1-\alpha_i)\,\alpha_j\,(y_i+y_j)\,x}{\alpha_i\,(1-\alpha_j)\,(x_i+x_j) + (\alpha_j-\alpha_i)\,x}\ .$$

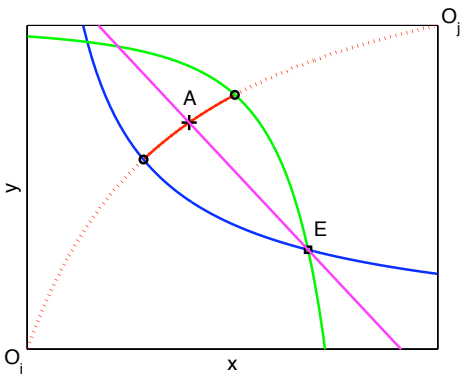

Figure 3.1 Edgeworth Box: The horizontal axis represents the amount of good x and the vertical axis represents the amount of good y of participant A_i. The point $O_j = (x_i + x_j, y_i + y_j)$ is the vertex opposite to the origin. The horizontal and vertical lines starting at the opposite vertex are the axes representing the amounts of good x and y, respectively, of participant A_j. Participant A_i has the minimum possible allocation at the origin $O_i = (0,0)$ and has a maximum allocation at O_j. The indifference curve for participant A_i is the blue convex curve and for participant A_j is the green concave curve. The red dots connecting O_i to O_j is the contract curve and the red curve connecting the two indifference curves is the core. The slope of the pink segment line is the bilateral equilibrium price. The interception point (A) of the core with the pink segment line determines the new allocations and the square (E) marks the initial endowments. Reproduced from [58].

The core is the subset of the contract curve such that the utility of both participants is greater than or equal to their initial utility. The bilateral equilibrium price b is the Walrasian price restricted to the market with these two participants. I.e. when $N = 2$ in Equation 3.3 we obtain

$$b = \frac{\alpha_i y_i + \alpha_j y_j}{(1-\alpha_i)x_i + (1-\alpha_j)x_j} \,.$$

The bilateral price determines a segment of allocations that pass through the point corresponding to the initial endowments. The interception of this segment with the core determines the new allocations: the bilateral equilibrium $(\hat{x}_i, \hat{y}_i) = (\alpha_i (x_i + y_i/b), (1-\alpha_i)(y_i + b x_i))$ of the two participants.

3.3 Duality

We follow the concept of statistical *duality* introduced in the previous work by Pinto et al [58]. It guarantees that the prices observed in the random matching Edgeworthian economy coincide in expectation with those of the Walrasian economy. This happens because it imposes on each participant a dual (mirror) participant with opposite preference, and symmetric initial endowments. We begin by presenting the concept of duality and some properties of the model.
Let participants A and B be participants from $\mathscr{A}$, having utility functions of the Cobb-Douglas type. Participant A has preference $\alpha \in (0,1)$ and endowments $(x,y) \in \mathbb{R}^2_+$, and participant B has preference $\alpha' \in (0,1)$ and endowments $(x',y') \in \mathbb{R}^2_+$. We will abbreviate as $A = (\alpha,x,y)$ and $B = (\alpha',x',y')$. Let $p \in \mathbb{R}_+$.

Definition 2. *The* p-dual *of participant A, denoted $\bar{A}$, has preference $\bar{\alpha} = 1-\alpha$ and endowments $(\bar{x},\bar{y}) = \left(\frac{y}{p}, px\right) \in \mathbb{R}^2_+$*

Proposition 1. *The p-dual participant $\bar{A}$ will have a logarithm of its marginal rate of substitution that is symmetric with respect to* $\ln p$ *to the marginal rate of substitution of participant A:*

$$\ln(MRS(\bar{A})) = 2\ln p - \ln(MRS(A)) \ .$$

Hence, when participant A has a *MRS* equal to (or above or below) p, her dual will have a *MRS* equal to (or below or above) p.

Proof. The utility of $\bar{A}$ is given by

$$u(\bar{A}) = \left(\frac{y}{p}\right)^{1-\alpha} (px)^{\alpha}.$$

Hence,

$$u\left(\bar{A}\right) = p^{2\alpha-1}u(A) \ .$$

Computing the marginal rate of substitution of $\bar{A}$, we have

$$MRS(\bar{A}) = \frac{\frac{\partial u(\bar{A})}{\partial \bar{x}}}{\frac{\partial u(\bar{A})}{\partial \bar{y}}} \ .$$

The derivatives are

$$\frac{\partial u(\bar{A})}{\partial \bar{x}} = \bar{\alpha}\frac{u(\bar{A})}{\bar{x}} \quad \text{and} \quad \frac{\partial u(\bar{A})}{\partial \bar{y}} = (1-\bar{\alpha})\frac{u(\bar{A})}{\bar{y}} \ .$$

Hence, the marginal rate of substitution is

$$MRS(\bar{A}) = \frac{\bar{\alpha}\bar{y}}{(1-\bar{\alpha})\bar{x}} \ .$$

Using Definition 2, we get

$$MRS(\bar{A}) = \frac{(1-\alpha)px}{\alpha y/p} = \frac{p^2}{MRS(A)} .$$

The proof is concluded by applying logarithm to the last equation.

□

After participants A and B meet, their new allocations will move them to, respectively, $A^{tr} = (\alpha, x+\Delta x, y-\Delta y)$ and $B^{tr} = (\alpha', x'-\Delta x, y'+\Delta y)$, with $(\Delta x, \Delta y) \in \mathbb{R}^2$.
We note that either Δx and Δy have the same sign, i.e. $\Delta x \Delta y > 0$; or they are both zero when trade does not occur, i.e. $(\Delta x, \Delta y) = (0,0)$. Moreover, in the former case we interpret as participant A giving an amount Δy and receiving in exchange an amount Δx, if both Δx and Δy are positive; or the reverse, if both are negative. Furthermore, we define the trading price as

$$p_{tr} = \frac{\Delta y}{\Delta x} .$$

Definition 3. *The* eye-shaped *region of possible trade between participants A and B in* $\mathbb{R}^2_+$ *is the region of trade that does not decrease the utility of either participant:*

$$eye(A,B) = \{(\Delta x, \Delta y) \in \mathbb{R}^2 : u\left(A^{tr}\right) \geq u(A) \text{ and } u\left(B^{tr}\right) \geq u(B)\} .$$

The eye-shaped region is the intersection between the area above the indifference curve of participant A_i in Figure 3.1 and the area below the indifference curve of participant A_j. Let $p > 0$, let $(\Delta x, \Delta y)$ be in $\mathbb{R}^2$, and let the map $m_p : \mathbb{R}^2 \to \mathbb{R}^2$ be $m_p(\Delta x, \Delta y) = \left(\frac{\Delta y}{p}, p\Delta x\right)$. Let A and B be two participants from $\mathscr{A}$. Lemma 1 states that the eye-shaped region of the p-dual participants $\bar{A}$ and $\bar{B}$ is obtained by applying the map m_p to the eye-shaped region of A and B.

Lemma 1. *Let A and B be two participants from* $\mathscr{A}$*. The eye-shaped region of the p-duals of A and B is* $m_p\left(eye(A,B)\right)$*, i.e. for all* $(\Delta x, \Delta y) \in eye(A,B)$

$$m_p(\Delta x, \Delta y) = \left(\frac{\Delta y}{p}, p\Delta x\right) \in eye(\bar{A}, \bar{B}) .$$

Proof. We need to prove that for all $(\Delta x, \Delta y) \in eye(A,B)$ which does not decrease the utility of neither A nor B, will imply that the trade $m_p(\Delta x, \Delta y)$ does not decrease the utility of neither $\bar{A}$ nor $\bar{B}$.
By Definition 3, if $(\Delta x, \Delta y) \in eye(A,B)$ then

$$u\left(A^{tr}\right) \geq u(A) . \tag{3.4}$$

Therefore

$$(x+\Delta x)^{\alpha} (y-\Delta y)^{1-\alpha} \geq x^{\alpha} y^{1-\alpha} .$$

After $\bar{A}$ and $\bar{B}$ trade, the utility of $\bar{A}^{tr}$ is

$$u\left(\bar{A}^{tr}\right) = u\left(1-\alpha, \frac{y-\Delta y}{p}, p(x+\Delta x)\right) .$$

Applying the Cobb-Douglas utility function

$$u\left(\bar{A}^{tr}\right) = \left(\frac{y-\Delta y}{p}\right)^{1-\alpha} (p(x+\Delta x))^{\alpha} .$$

The inequality in Equation 3.4 implies that

$$u\left(\bar{A}^{tr}\right) \geq \left(\frac{y}{p}\right)^{1-\alpha} (px)^{\alpha} .$$

Hence,

$$u\left(\bar{A}^{tr}\right) \geq u\left(\bar{A}\right) .$$

A similar result is obtained for the p-dual of participant B. Therefore, if $(\Delta x, \Delta y) \in eye(A,B)$, $\left(\frac{\Delta y}{p}, p\Delta x\right)$ must belong to $eye(\bar{A},\bar{B})$.

□

The p-statistical duality

Consider a collection of participants $\mathscr{A}$ with preferences α_i, and endowments (x_i, y_i). The probability density function $f(\alpha_i, x_i, y_i)$, on the (α_i, x_i, y_i) space $(0,1) \times \mathbb{R}^2_+$, provides the probability that a participant A_i has preferences in $(\alpha_i, \alpha_i + d\alpha_i) \times (x_i, x_i + dx) \times (y_i, y_i + dy)$. We assume that the probability distribution has compact support, and the support in (x_i, y_i) is bounded away from zero.

P-statistical duality probability density function

We will use the p-statistical duality of the market as defined by Pinto *et al.* [58]. Let $p \in \mathbb{R}_+$.

Definition 4 (Pinto *et al.* [58])**.** *A market satisfies the* **p - statistical market duality** *condition if for any participant $A \in \mathscr{A}$ and its dual $\bar{A} \in \mathscr{A}$ the probability density function has the symmetry property*

$$f(A) = f(\bar{A}) .$$

The p-statistical duality property means that each participant with characteristics (α, x, y) has a mirror participant with characteristics $(1-\alpha, y/p, px)$ with the same probability density under f. An example of functions that satisfy the p-statistical duality are the class of functions $f(\alpha, x, y)$ of the form $f_1(\alpha) f_2(x,y)$ with the property that f_1 is symmetric on preferences $f_1(\alpha) = f_1(1-\alpha)$ and f_2 is p-symmetric in endowments $f_2(x,y) = f_2(y/p, px)$.

Let us denote the initial random choice of participants by $\omega_{\mathscr{A}}$, and define the initial choice of participants that will be able to trade by the random variable $\mathscr{A}(\omega_{\mathscr{A}}) = \{A_1, A_2, \cdots A_N\}$.

Proposition 2. *If a market satisfies the p-statistical market duality, the expectation of logarithm of the marginal rate of substitution over all possible distributions of participants $A(\omega_{\mathscr{A}})$ is p:*

$$\mathbb{E}(\ln MRS(A)) = \ln p \ .$$

Proof. Since any participant and its dual have the same probability of being selected, and the market duality holds, then $\mathbb{E}(\ln MRS(\bar{A}))$ must be equal to $\mathbb{E}(\ln MRS(A))$. The proof is concluded by applying Proposition 1 and reorganizing the terms.

□

3.4 Deviating from the bilateral equilibrium

We will consider a model where participants meet in pairs and have a limited view of the world. The pair might be unable to choose rationally their optimal trade that will bring them to their bilateral equilibrium. They are bounded to select trades that do not decrease their respective utility. This model is inspired by the constrained zero-intelligent participants as described by Gode and Sunder [39]. Each participant is limited by her current utility, but other than that, they may have zero intelligence: their choice of trade may be random. Hence, after trade, the participants may end in a point inside the eye-shaped region in figure 3.1 that is different from their bilateral equilibrium (point A).
Besides the duality in the participants, in this chapter we also assume that participants' choice of trade has a symmetric distribution. The p-statistical trading duality means that given a pair and its dual, it is as likely that the pair ends up with some allocations as is likely that its dual end up with the mirror allocations.

Definition 5. *We say that the trade satisfies the* **p-statistical trading duality** *if the probability density function q has the symmetry property, for all A and B and for all $(\Delta x, \Delta y) \in eye(A,B)$,*

$$q_{A,B}(\Delta x, \Delta y) = q_{\bar{A},\bar{B}}\left(\frac{\Delta y}{p}, p\Delta x\right).$$

Remark 1. *If trade occurs and it follows the p-statistical trading duality, the logarithm of the price of a trade done between participants A and B is* log*-p-symmetric to the logarithm of price of the p-dual trade done between their p-duals $\bar{A}$ and $\bar{B}$, even if q depends explicitly on the past and on which participants are being selected. Therefore, we have that $\ln \bar{p_{tr}} = 2\ln p - \ln p_{tr}$. In particular, if A and B trade at the price $p_{tr} = p$, then their duals will also trade at price p.*

Proof.

$$\ln\left(\frac{p\Delta x}{\frac{\Delta y}{p}}\right) = \ln\left(p^2\frac{\Delta x}{\Delta y}\right) = 2\ln p - \ln(p_{tr}) .$$

□

Using these properties, we can expand the result by Pinto *et al.* [58] to the case where trade may deviate form the bilateral equilibrium as long as we impose that the duals trade following a properly symmetric distribution.
In order to prove our results, we need to assume that there is at least one participant that has a positive probability of trading with any other participant. Our choice of f and q will be such that it exists a value $\varepsilon > 0$ such that Any participant B that we may chose will determine a region τ_ε inside of $eye(A,B)$, where trade increases the utility of at least one of the participants with τ_ε being defined as

$$\tau_\varepsilon = \{(\Delta x, \Delta y) \in eye(A,B) : u\left(A^{tr}\right) u\left(B^{tr}\right) > (1+\varepsilon)u(A)u(B)\}.$$

The region τ_ε is such that there is a sufficiently high probability of A and B trading:

$$Pr(A,B) \geq \int_{\tau_\varepsilon} q_{A,B}(\Delta x, \Delta y) > \varepsilon \ .$$

Remark 2. *Since participant A is "willing" to trade with everyone else, we could use as a simplified restriction a condition applied to B:*

$$\tau_\varepsilon = \{(\Delta x, \Delta y) \in eye(A,B) : u\left(B^{tr}\right) > (1+\varepsilon)u(B)\} \ .$$

We note that either definitions of τ_ε do not contain the region ε-near $(0,0)$, where no trade occurs and will not contain the "opposite" corner of the eye-shaped region, the region ε-near the other interception of the two indifference curves. One possible choice of q that follows this assumptions is q being the uniform distribution inside the eye-shaped region and zero outside of it. Other possibility would be to consider that q is such that the bilateral equilibrium is chosen with probability 1, that would correspond to the original model by Pinto *et al.* [58]. Another example would be to select a point in the core, as presented in [32] by Ferreira *et al.*, that is as a function of a selfishness type, also called bargaining skill or greediness type, that depends only on the participants chosen but not on their endowments, preferences, or history.

3.5 Random matching market

As in Pinto *et al.* [58], we consider a market where N participants are picked at random according to a given probability distribution f which satisfies the p-statistical duality. Each participant A_i has preference α_i and initial endowments $(x_i(0), y_i(0))$. Let $p(\omega_{\mathscr{A}})$ be the Walrasian equilibrium price of the market for this collection $\mathscr{A}(\omega_{\mathscr{A}})$ of N participants, and $\mathbb{E}[\ln(p\omega_{\mathscr{A}})]$ be the expectation of the logarithm of the Walrasian equilibrium price $p(\omega_{\mathscr{A}})$ computed with respect to the initial consumption bundles of the participants over all the initial collections $\mathscr{A}(\omega_{\mathscr{A}})$ of N participants. At each time instant t, a random pair i, j will be matched, with all pairs having the same probability of being selected. After t trades participants A_i and A_j have consumption bundles $(x_i(t), y_i(t))$ and $(x_j(t), y_j(t))$ respectively. At this instant, the pair i, j trades $(\Delta x(t), \Delta y(t)) \in eye(i, j) \subset \mathbb{R}^2$, according to the probability density function q that follows the p-statistical trading duality. After these trades we have that

$$x_i(t+1) = x_i(t) + \Delta x(t), \;\; y_i(t+1) = y_i(t) - \Delta y(t),$$
$$x_j(t+1) = x_j(t) - \Delta x(t), \;\; y_j(t+1) = y_j(t) + \Delta y(t),$$

with trade either occurring at a price $p(t) = \frac{\Delta_y}{\Delta_x} > 0$, or they decide not to exchange goods, thus $\Delta x = \Delta y = 0$. The consumption bundles of all the other participants $A_k \neq A_i, A_j$ remain unchanged, i.e. $(x_k(t+1), y_k(t+1)) = (x_k(t), y_k(t))$.

Let $\bar{\omega}_r$ be the infinite sequence of participants selected to trade at each instant in time. The variable $\bar{\omega}_r = (\bar{\omega}_r(1), \bar{\omega}_r(2), \cdots)$, where $\bar{\omega}_r(t) = (A_i(t), A_j(t))$, includes information about the pair $(A_i(t), A_j(t))$, $A_i(t) \neq A_j(t)$, corresponding to the pair of participants $(A_i(t), A_j(t))$ that have been randomly chosen to trade at time t. Let $\bar{\omega}_g$ be the infinite sequence $\bar{\omega}_g = (\bar{\omega}_g(1), \bar{\omega}_g(2), \cdots)$, where $\bar{\omega}_g(t) = (\Delta x(t), \Delta y(t))$ is the variable that includes the amounts traded. If both $\Delta x(t) > 0$ and $\Delta y(t) > 0$, we consider that participant $i(t)$ received $\Delta x(t)$ and gave $\Delta y(t)$ to participant $j(t)$. We note that each pair of quantities traded $(\Delta x, \Delta y)$ in $\bar{\omega}_g$ must belong to the eye-shaped region of the pair selected in that instant in time. Thus, the choice of possible values for $(\Delta x, \Delta y)$ is dependent on the history, in particular, it depends on $\omega_{\mathscr{A}}, \bar{\omega}_r$ and $\bar{\omega}_g$ restricted to that moment in time t. However, the probability density function q is only a function of the current state of the matched participants. Moreover, the participants choice of trade is not dependent on future outcomes of either participants. A full run of the game is the sequence $\omega_{\mathscr{A}} \bar{\omega}_r \bar{\omega}_g$ that is an initial choice of participants and an infinite sequence of random matchings and quantities exchanged. Following from our assumption, the participants will trade an infinite number of times. Since when $\Delta x(t) = 0$ and $\Delta y(t) = 0$ the market is unchanged; we will only consider the infinite subsequence of $\omega_{\mathscr{A}} \bar{\omega}_r \bar{\omega}_g$ such that the instants t at which no trade occurred $\Delta x(t) = 0$ and $\Delta y(t) = 0$ were removed. This subsequence will be denoted by $\omega_{\mathscr{A}} \omega_r \omega_g$. A finite time run of the game is the sequence $\omega_{\mathscr{A}} \omega_r \omega_g | t$, where $\omega_r \omega_g | t$ is the restriction of $\omega_r \omega_g$ for the first t random matches.

Let $p_{tr}(t, \omega_{\mathscr{A}} \omega_r \omega_g)$ be the trading price of the trade at time t determined by the finite run $\omega_{\mathscr{A}} \omega_r \omega_g | t$. Let $\mathbb{E}[\ln p_{tr}(t, \omega_{\mathscr{A}} \omega_r \omega_g)]$ be the expected value of the logarithm of $p_{tr}(t, \omega_{\mathscr{A}} \omega_r \omega_g)$ over all initial collections $\mathscr{A}(\omega_{\mathscr{A}})$ of participants and over the first t random meetings that trade amounts given by the first values t of the random variable ω_g. By Gale [36] the limiting price $p_{tr}(\omega_{\mathscr{A}} \omega_r \omega_g) =$

$\lim_{t\to\infty} p_{tr}(t,\omega_{\mathscr{A}}\omega_r\omega_g)$ exists almost surely and it is a random variable dependent on the sequence of matches of the participants and on the amounts traded. Let $\mathbb{E}[\ln p_{tr}(\omega_{\mathscr{A}}\omega_r\omega_g)]$ be the expected value of the logarithm of the limiting price $p(\omega_{\mathscr{A}}\omega_r\omega_g)$ when $t\to\infty$ over all the possible distributions of the participants $\mathscr{A}(\omega_{\mathscr{A}})$ and over all random matchings ω_r and over all possible trades ω_g. Expanding the results from Pinto *et al.* [58], we can prove that also in this market the duality determines the expectation of the logarithm of the Walrasian price of the initial market and the expectation of the logarithm of the trading price.

Theorem 3 (p-statistical duality fixed point)**.** *Assume a market G consisting of a finite number of participants, such that p-statistical market duality holds for the initial endowments and the p-statistical trading duality holds, then for all* $t\in\{1,2,\cdots,+\infty\}$

$$\mathbb{E}[\ln p_{tr}(t,\omega_{\mathscr{A}}\omega_r\omega_g)]=\mathbb{E}[\ln p_W(\omega_{\mathscr{A}})]=\ln p\ .$$

Furthermore,

$$\mathbb{E}[\ln p_{tr}(\omega_{\mathscr{A}}\omega_r\omega_g)]=\ln p\ .$$

In Theorem 3, the advantage of using the logarithm of the price is that the duals trade at the symmetric value with the same probability. In particular, Theorem 3 is a fixed point theorem for the expected value $\mathbb{E}[\ln p_{tr}(t,\omega_{\mathscr{A}}\omega_r\omega_g)]$ that is constant along time t. Furthermore, and similarly as stated in Pinto *et al* [58], this result holds even when $n(t)\geq 2$ participants trade simultaneously, and are selected with a non-uniform probability distribution, provided the trading duality condition and the trading assumptions 1 and 2 are properly adjusted.

Proof of Theorem 3

The proof follows the steps in Pinto *et al.* [58]. We begin by using Gale's Lemma which follows from Proposition 3, Chapter 1 of [36]. The proof that $\mathbb{E}[\ln p_W(\omega_A)]=\ln p$ follows directly from Pinto *et al.* [58]. We identify participant A_i at time t, hereafter denoted by $A_i(t)$, with $(\alpha_i,x_i(t),y_i(t))$ consisting of her preference and her amounts of goods x and y at time t. We write $\bar{A}_i(t)=A_j(t)$ to state that participant $A_i(t)$ at time t is *dual* to participant $A_j(t)$ at time t, i.e., when $\alpha_j=1-\alpha_i$, and $(x_i(t),y_i(t))=(y_j(t)/p,px_j(t))$.

Proof. Suppose that we have two initial sets of participants

$$\mathscr{A}=\{A_1,A_2,\cdots,A_N\},$$
$$\mathscr{B}=\{B_1,B_2\cdots,B_N\},$$

such that every participant $B_i=\bar{A}_i$ is the dual participant of A_i. Choose a run of the play ω_r. Let $(\Delta x,\Delta y)$ be the amounts traded by the pair (A_i,A_j) at time t, with either $\Delta x\Delta y\geq 0$ or $\Delta x=0$ and $\Delta y=0$, with probability density $q_{A_i,A_j}(\Delta x,\Delta y)$. If trade occurred, its price is given by

$$p_{tr}(t,\omega_{\mathscr{A}}\omega_r\omega_g)=p_{ij}(t)=\frac{\Delta y}{\Delta x}\ .$$

Our duality conditions mean that the dual pair $(\bar{A}_i, \bar{A}_j)$ will trade the dual quantities $\Delta\bar{x} = \frac{\Delta y}{p}$, $\Delta\bar{y} = p\Delta x$ with the same probability density

$$\bar{q}_{\bar{A}_i(t),\bar{A}_j(t)}(\Delta\bar{x}, \Delta\bar{y}) = q_{A_i(t),A_j(t)}(\Delta x, \Delta y) \ .$$

Hence, the dual trade occurred at price

$$\frac{\Delta\bar{y}}{\Delta\bar{x}} = \frac{p\Delta x}{\frac{\Delta y}{p}} = \frac{p^2}{p_{ij}}(t) \ .$$

After the trade at time $t+1$, the probability density of the participants (A_i, A_j) having allocations

$$\begin{aligned} x_i(t+1) &= x_i(t) + \Delta x = x_i^*, \\ y_i(t+1) &= y_i(t) - \Delta y = y_i^*, \end{aligned}$$

is equal to the probability density of the dual pair of participants $(B_i, B_j) = (\bar{A}_i, \bar{A}_j)$ having the dual allocations

$$\begin{aligned} \bar{x}_i(t+1) &= \bar{x}_i + \Delta\bar{x} = \bar{x}_i^* = \frac{y_i^*}{p}, \\ \bar{y}_i(t+1) &= \bar{y}_i - \Delta\bar{y} = \bar{y}_i^* = px_i^*, \end{aligned} \tag{3.5}$$

with similar expressions for j.
We observe from Equation 3.5 that if $\mathscr{A}$ and $\mathscr{B}$ follow the same sequence ω_r, we must have that $B_i(t) = \bar{A}_i(t)$ for all i. Therefore, the trading prices are equal

$$p\left(\bar{A}_i(t+1), \bar{A}_j(t+1)\right) = p\left(B_i(t+1), B_j(t+1)\right). \tag{3.6}$$

Which means that the random dynamical system defined by Equation (3.5) is equivariant under the duality transformations.
Therefore, the market duality and the trading duality allow us to map each run of the economy $\omega = \omega_{\mathscr{A}}\omega_r\omega_g$ to a dual run $\bar{\omega} = \bar{\omega}_{\mathscr{A}}\bar{\omega}_r\bar{\omega}_g$ with the same probability $Pr(\omega_{\mathscr{A}}\omega_r\omega_g|t) = Pr(\bar{\omega}_{\mathscr{A}}\bar{\omega}_r\bar{\omega}_g|t)$. Therefore, by Equation (3.6), the statistical duality is invariant over time. Again, by Equation (3.6), we obtain that

$$\begin{aligned} &\ln p_{ij}(t, \omega_A\omega_r\omega_g) + \ln \bar{p}_{ij}(t, \omega_A\omega_r\omega_g) \\ &= \ln p_{ij}(t, \omega_A\omega_r\omega_g) + \ln \frac{p^2}{p_{ij}(t, \omega_A\omega_r\omega_g)} \\ &= \ln p_{ij}(t, \omega_A\omega_r\omega_g) + 2\ln p - \ln p_{ij}(t, \omega_A\omega_r\omega_g) \\ &= 2\ln p, \end{aligned} \tag{3.7}$$

which implies, by statistical duality, that

$$\mathbb{E}[\ln p_{tr}(t, \omega)] = \ln p, \ \forall\, t \in \mathbb{N} \ .$$

We note that Equation (3.5) implies the invariance of the expectation of the marginal rate of substitution for dual pairings. Let us assume that $MRS(\bar{A}_i) = p^2/MRS(A_i)$ holds at time t. Then, at time $t+1$, after pair i and j traded $(\Delta x, \Delta y) \in \mathbb{R}^2_+$ we have that

$$MRS(A_i)(t+1) = \frac{\alpha_i(y_i - \Delta y)}{(1-\alpha_i)(x_i + \Delta x)} \ .$$

Her p-dual will have her MRS symmetric with respect to $\ln p$, hence, the expected value over ω_A, ω_r and ω_g of $\ln MRS$ will be fixed over time.
Now, let us consider $t = \infty$. There exists a constant $K \geq 0$ such that for all t, we have that

$$| \ln p_{tr}(t, \omega_{\mathscr{A}} \omega_r \omega_g) \ | \leq K \text{ almost surely,}$$

$$\text{provided } |\ln p_{tr}(t, \omega_A \omega_r \omega_g)| < K \text{ for some } K = 2 \ln k,$$

where k is a constant.
The boundedness of the price follows from the assumption that all the distribution of endowments for the participants has compact support which is bounded away from 0 for all t. Then by a direct application of Lebesgue's dominated convergence theorem we have that

$$\mathbb{E}\left[\lim_{t\to\infty} \ln p_{tr}(t, \omega_{\mathscr{A}} \omega_r \omega_g)\right] = \lim_{t\to\infty} \mathbb{E}\left[\ln p(t, \omega_{\mathscr{A}} \omega_r \omega_g)\right] = \mathbb{E}[\ln p(\omega_{\mathscr{A}} \omega_r \omega_g)],$$

from which follows that

$$\mathbb{E}[\ln p_{tr}(\omega_{\mathscr{A}} \omega_r \omega_g)] = \mathbb{E}[\ln p_W(\omega_{\mathscr{A}})] = \ln p \ .$$

□

Chapter 4

General Conclusion

In this work we applied concepts of Dynamical Systems and Game Theory to immunology, in Chapter 2, on one hand and to random exchange economies on the other hand, in Chapter 3.

We studied a model of immune response by $CD4^+$ T cells with $CD4^+$ Tregs with asymmetric death rates. We expanded previous results [21–25, 54, 57] by presenting explicit formulas that give the relation at equilibria (stable or unstable) between the concentration x of T cells, Tregs, IL-2 cytokines and the antigenic stimulation b of T cells. The relation between the concentration of T cells and their antigenic stimulation is a hysteresis. Between the two thresholds b_L and b_H of antigenic stimulation of T cells, the hysteresis has a bounded region of bistability (containing also one unstable equilibrium). For the default values of our parameters, we observed an unfold of the hysteresis for $d_{T^*}/d_T = 1.71$. When we considered a linear tuning between the antigenic stimulations of T cells and Tregs, we observed a change in behavior with the slope parameter m. When we changed the slope parameter we found the appearance of an isola, a saddle-node equilibria. The isola reaches the hysteresis and a transcritical bifurcation is present when $m = 4.1 \times 10^{-2}$. Possible extensions of our immunology model can include explicit equations for the presence of memory T cells and memory Tregs. Additionally, we may consider the presence of different lines of T cells responding to different antigens, at different levels.

We studied prices in a model under the scope of General Equilibrium Theory. We looked at a random exchange economy with two goods, where the participants' preferences are characterized by the Cobb-Douglas utility function. We expanded the results in Pinto *et al.* [58], allowing trade to deviate from the bilateral equilibrium. We introduced the concept of p-statistical trading duality, and also discussed the deviation form the bilateral equilibrium price through the introduction of participants that may not be fully rational. We proved that under symmetry conditions, the trading price is related with the Walrasian price of the initial market. As future work, it would be interesting to generalize our study to other utility functions than the Cobb-Douglas, with the proper generalization of the duality condition, and to extend such results for the case of financial markets.

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